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Abstract. As recently observed by Bazhlekova and Dimovski [2], the n-th derivative of the 2-parametric Mittag-Leffler function gives a 3-parametric Mittag-Leffler function, known as the Prabhakar function. Following the analogy, the n-th derivatives of the Bessel–Maitland functions are obtained, and it turns out they are expressed in terms of the generalized Bessel–Maitland functions with 3 indices. Further, some special cases of the fractional order Riemann–Liouville derivatives and integrals of the Bessel–Maitland functions are calculated and interesting relations are proved.

INTRODUCTION

The classical Bessel functions of the first kind are defined by the series

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k!\, \Gamma(k+\nu+1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0];\ \nu \in \mathbb{C}, \tag{1}$$

and the Bessel–Clifford functions, which are entire functions closely related with (1), are defined by the power series

$$C_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z)^k}{k!\, \Gamma(k+\nu+1)}, \quad z \in \mathbb{C};\ \nu \in \mathbb{C}. \tag{2}$$

Because of their intensive use and various applications both kind of functions have many generalizations with more indices (parameters). Here we consider the generalizations of the Bessel and Bessel–Clifford functions with two and three indices. We begin with the entire functions

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\, \Gamma(\mu k+\nu+1)}, \quad z \in \mathbb{C};\ \nu \in \mathbb{C} \ \text{ and } \ \mu > -1, \tag{3}$$

introduced by the British mathematician Sir Edward Maitland Wright [12] and known in the literature as Bessel–Maitland functions (named after his second name). Wright firstly introduced them for $\mu > 0$ and on a later stage, studying the asymptotic behavior of the Bessel–Maitland functions for large values of $|z|$, he extended their definition for $\mu > -1$ ([13], see also [5] and [6]). Let us just mention, that for $\mu = 1$ the Bessel–Maitland function becomes Bessel–Clifford function, i.e.

$$C_\nu(z) = J_\nu^1(z)$$

We also consider the generalized Bessel–Maitland (or Bessel–Wright) functions with 3 parameters (introduced by R.S. Pathak [10]):

$$J_{\nu,\sigma}^\mu(z) = (z/2)^{\nu+2\sigma} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(k+\sigma+1)\Gamma(\mu k+\sigma+\nu+1)}, z \in \mathbb{C} \setminus (-\infty, 0]; \quad \nu, \sigma \in \mathbb{C}, \quad \mu > 0. \tag{4}$$

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It is easily to see there is a simple relation between the two-index Bessel–Maitland functions $J_\nu^\mu(z)$ and the generalized Bessel–Maitland functions with 3 indices, namely

$$J_\nu^\mu(z) = z^{-\nu/2}\, J_{\nu,0}^\mu(2\sqrt{z}), \quad z \in \mathbb{C} \setminus (-\infty, 0];\ \nu \in \mathbb{C},\ \mu > 0, \tag{5}$$

and in particular, for $\mu = 1$ the relation is between C_ν and $J_{\nu,0}^1 = J_\nu$, i.e.

$$C_\nu(z) = z^{-\nu/2} J_\nu(2\sqrt{z}), \quad z \in \mathbb{C} \setminus (-\infty, 0],\ \nu \in \mathbb{C}. \tag{6}$$

Studying the properties of the functions, mentioned above, integrals and derivatives of an arbitrary order of the functions (3) are found. As a result, different interesting relations between these integrals and derivatives and functions of the kind (4) are obtained. The corresponding special cases for the Bessel–Clifford functions (2) merely follow as corollaries, taking $\mu = 1$. Such type of results are also obtained for other special functions of Fractional Calculus, for more general functions see e.g. [3] and [9], for the Mittag-Leffler functions see [8]. Various useful results for the generalized fractional calculus operators of special functions are given in the interesting paper [4].

INTEGER ORDER DERIVATIVES OF THE BESSEL–MAITLAND FUNCTIONS

Recently, it has been obtained by Bazhlekova and Dimovski [2] that the n-th derivative of the 2-parametric Mittag-Leffler function gives a 3-parametric Mittag-Leffler function, introduced by Prabhakar (up to a constant), namely

$$E_{\alpha,\beta}^{(n)}(z) = n!\, E_{\alpha,\beta+n\alpha}^{n+1}(z). \tag{7}$$

Following the analogy, the n-th derivative of the two-index Bessel–Maitland function (3), which is closely connected with the Mittag-Leffler function, is calculated. It turns out that this integer order derivative is expressed by a function of the same kind, up to a constant. Further, due to the relation (5), the result can be reduced to the 3-parametric generalized Bessel–Maitland function (4), up to a power function.

Theorem 1 *Let z be a complex variable and let μ, ν be the parameters, satisfying the conditions $\nu \in \mathbb{C}$, $\mu > -1$. Then the following equalities hold true for all the values of $n \in \mathbb{N}_0$:*

$$D^n \left[J_\nu^\mu(z)\right] = \frac{d^n}{dz^n} \left[J_\nu^\mu(z)\right] = (-1)^n J_{\nu+n\mu}^\mu(z), \quad z \in \mathbb{C}, \tag{8}$$

and

$$D^n \left[J_\nu^\mu(z)\right] = z^{-(\nu+n)/2}\, J_{\nu+n,-n}^\mu(2\sqrt{z}), \quad z \in \mathbb{C} \setminus (-\infty, 0]; \quad \mu > 0. \tag{9}$$

Proof Bearing in mind that for $n \in \mathbb{N}$

$$D^n\left(z^k\right) = \frac{\Gamma(k+1)}{\Gamma(k-n+1)}\, z^{k-n},$$

and taking a term-by-term differentiation under the summation sign (which is possible in accordance with the uniform convergence of the series (3) in any compact subset of $\mathbb{C}$ and the differentiability of the power function) we obtain

$$D^n \left[J_\nu^\mu(z)\right] = \sum_{k=n}^{\infty} \frac{(-1)^k z^{k-n}}{\Gamma(k-n+1)\Gamma(\mu k+\nu+1)} \tag{10}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{\Gamma(k+1)\Gamma(\mu k+n\mu+\nu+1)} = (-1)^n J_{\nu+n\mu}^\mu(z),$$

that verifies (8) for $n \in \mathbb{N}$. In the case $n = 0$, we have

$$D^0 \left[J_\nu^\mu(z)\right] = J_\nu^\mu(z) = (-1)^0 J_\nu^\mu(z) = (-1)^0 J_{\nu+0\mu}^\mu(z),$$

therefore (8) is also fulfilled. Before proving (9) we firstly note that the formula (10) can also be written in the next form

$$D^n \left[J_\nu^\mu(z)\right] = z^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{\Gamma(k-n+1)\Gamma(\mu k+\nu+1)} \tag{11}$$

with $1/\Gamma(k-n+1) = 0$ for $k = 0, 1, \ldots, n-1$, which is equivalent to

$$D^n \left[J_\nu^\mu(z) \right] = z^{-(\nu+n)/2} (\sqrt{z})^{\nu+n-2n} \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{z})^{2k}}{\Gamma(k-n+1)\Gamma(\mu k+\nu+1)} = z^{-(\nu+n)/2} J_{\nu+n,-n}^\mu (2\sqrt{z}), \tag{12}$$

for, $\mu > 0$. Thus the correctness of (9) immediately follows. □

In particular, taking $\mu = 1$, Theorem 1 gives results referring to the classical Bessel–Clifford functions, i.e. we produce the following corollary.

Corollary 1 *Let the variable $z \in \mathbb{C}$ and the parameter $\nu \in \mathbb{C}$. Then the following equalities hold true for all the values of $n \in \mathbb{N}_0$:*

$$D^n C_\nu(z) = [C_\nu(z)]^{(n)} = (-1)^n C_{\nu+n}(z), \tag{13}$$

and

$$D^n C_\nu(z) = [C_\nu(z)]^{(n)} = z^{-(\nu+n)/2} J_{\nu+n,-n}^1 (2\sqrt{z}) \quad (z \in \mathbb{C} \setminus (-\infty, 0]). \tag{14}$$

Proof Putting $\mu = 1$, the function $J_\nu^\mu(z)$ becomes $C_\nu(z)$. Therefore the following logical conclusions can be deduced from (8) and (9):

$$D^n C_\nu(z) = [C_\nu(z)]^{(n)} = \left[J_\nu^1(z) \right]^{(n)} = (-1)^n J_{\nu+n}^1(z) = (-1)^n C_{\nu+n}(z)$$

and respectively

$$D^n C_\nu(z) = [C_\nu(z)]^{(n)} = \left[J_\nu^1(z) \right]^{(n)} = z^{-(\nu+n)/2} J_{\nu+n,-n}^1 (2\sqrt{z}),$$

which complete the proof. □

FRACTIONAL RIEMANN–LIOUVILLE INTEGRALS AND DERIVATIVES

The most popular definition for integration of arbitrary (i.e. not obligatorily integer) order $\lambda \in \mathbb{C}$ ($Re(\lambda) > 0$) is the *Riemann–Liouville* (*R-L*) *fractional integral* [11]

$$R^\lambda f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} f(t) dt = \frac{z^\lambda}{\Gamma(\lambda)} \int_0^1 (1-\tau)^{\lambda-1} f(z\tau) d\tau \,. \tag{15}$$

The corresponding *Riemann–Liouville fractional derivative* of order is defined as a composition of a derivative of integer order and an integral of fractional order of the form (15), namely:

$$D^\lambda f(z) := D^n R^{n-\lambda} f(z), \tag{16}$$

where $n := [Re(\lambda)] + 1 > Re(\lambda)$, $[Re(\lambda)]$ = integer part of $Re(\lambda)$.

For the theory and applications of the fractional Riemann–Liouville integral and derivative, as well as of other integral and differential operators of fractional calculus (FC), see the FC encyclopedia [11] and monograph [3], for the evolution and development of FC, see [7]. For miscellaneous useful applications of a number operators of FC, see also the recent survey paper [1].

In this section we find the R-L fractional integrals and derivatives (15) and (16) of the two-index Bessel–Maitland function (3). Moreover we establish the connections between these integrals and derivatives and the three-index Bessel–Maitland function (4). The corresponding elementary assertions for the Bessel–Clifford functions (2) follow in the simplest case $\mu = 1$.

Theorem 2 *Let the variable $z \in \mathbb{C}$, the parameters $\lambda, \nu \in \mathbb{C}$, $Re(\lambda) > 0$ and $\mu > 0$. Then the Riemann–Liouville fractional derivative $D^\lambda[J_\nu^\mu(z)]$ exists and*

$$D^\lambda[J_\nu^\mu(z)] = z^{-(\lambda+\nu)/2} J_{\nu+\lambda,-\lambda}^\mu (2\sqrt{z}) \quad (|\arg z| < \pi). \tag{17}$$

Proof Having in mind the well-known formula [11]

$$D^{\lambda}\left(z^{p}\right)=\frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\,z^{p-\lambda}\quad\text{for}\quad Re(p)>-1, \tag{18}$$

one can write down

$$D^{\lambda}\left(z^{k}\right)=\frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)}\,z^{k-\lambda},\quad k=0,1,2,\ldots,$$

and exchanging again the orders of differentiation and summation (justified by the integrability and differentiability of the power function for $|\arg z|<\pi$ and uniform convergence of the series in any compact subset of this set), we obtain

$$D^{\lambda}\left[J_{\nu}^{\mu}(z)\right]=z^{-\lambda}\sum_{k=0}^{\infty}\frac{(-1)^{k}z^{k}}{\Gamma(k-\lambda+1)\Gamma(\mu k+\nu+1)} \tag{19}$$

$$=z^{-(\nu+\lambda)/2}\,(\sqrt{z})^{\nu+\lambda-2\lambda}\sum_{k=0}^{\infty}\frac{(-1)^{k}(\sqrt{z})^{2k}}{\Gamma(k-\lambda+1)\Gamma(\mu k+\nu+1)}=z^{-(\nu+\lambda)/2}\,J_{\nu+\lambda,-\lambda}^{\mu}(2\sqrt{z}),$$

which completes the proof. □

Corollary 2 *Let z be a complex variable, the parameters $\lambda,\nu\in\mathbb{C}$ and $Re(\lambda)>0$. Then the Riemann–Liouville fractional derivative $D^{\lambda}[C_{\nu}(z)]$ exists and*

$$D^{\lambda}[C_{\nu}(z)]=z^{-(\lambda+\nu)/2}\,J_{\nu+\lambda,-\lambda}^{1}(2\sqrt{z})\quad(|\arg z|<\pi). \tag{20}$$

Proof Taking $\mu=1$, the relation (17) immediately implies

$$D^{\lambda}[C_{\nu}(z)]=D^{\lambda}[J_{\nu}^{1}(z)]=z^{-(\lambda+\nu)/2}\,J_{\nu+\lambda,-\lambda}^{1}(2\sqrt{z})\quad(|\arg z|<\pi),$$

which confirms (20). □

Theorem 3 *Let the variable $z\in\mathbb{C}$, the parameters $\lambda,\nu,\omega\in\mathbb{C}$, $Re(\lambda)>0$ and $\mu>0$. Then the Riemann–Liouville fractional integral $R^{\lambda}[J_{\nu}^{\mu}(\omega z)]$ exists and the following relations hold true*

$$R^{\lambda}[J_{\nu}^{\mu}(\omega z)]=\frac{z^{(\lambda-\nu)/2}}{\omega^{(\lambda+\nu)/2}}\,J_{\nu-\lambda,\lambda}^{\mu}(2\sqrt{\omega z})\quad(|\arg z|<\pi,\ \omega>0), \tag{21}$$

and (*in particular*)

$$R^{\lambda}[J_{\nu}^{\mu}(z)]=z^{(\lambda-\nu)/2}\,J_{\nu-\lambda,\lambda}^{\mu}(2\sqrt{z})\quad(|\arg z|<\pi). \tag{22}$$

Proof Putting, for the sake of brevity,

$$f(z)=J_{\nu}^{\mu}(\omega z),$$

we have

$$f(z\tau)=J_{\nu}^{\mu}(\omega z\tau)=\sum_{k=0}^{\infty}\frac{(-\omega z)^{k}\,\tau^{k}}{k!\,\Gamma(\mu k+\nu+1)},$$

whence, using the second part of definition (15), we obtain consequently:

$$R^{\lambda}f(z)=\frac{z^{\lambda}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(-\omega z)^{k}}{k!\,\Gamma(\mu k+\nu+1)}\int_{0}^{1}(1-\tau)^{\lambda-1}\,\tau^{k}\,d\tau$$

$$=\frac{z^{\lambda}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(-\omega z)^{k}}{k!\,\Gamma(\mu k+\nu+1)}\frac{\Gamma(\lambda)\Gamma(k+1)}{\Gamma(k+\lambda+1)}=z^{\lambda}\sum_{k=0}^{\infty}\frac{(-\omega z)^{k}}{\Gamma(k+\lambda+1)\,\Gamma(\mu k+\nu+1)}.$$

Changing the orders of integration and summation is possible due to the integrability of the power function for $|\arg z| < \pi$ and uniform convergence of the series above in any compact subset of this set.

Now, in view of the identity

$$z^{\lambda} = \frac{z^{(\lambda-\nu)/2}}{\omega^{(\lambda+\nu)/2}} \left(\sqrt{\omega z}\right)^{\nu+\lambda} = \frac{z^{(\lambda-\nu)/2}}{\omega^{(\lambda+\nu)/2}} \left(\sqrt{\omega z}\right)^{\nu-\lambda+2\lambda},$$

we conclude that

$$R^{\lambda} f(z) = \frac{z^{(\lambda-\nu)/2}}{\omega^{(\lambda+\nu)/2}} \left(\sqrt{\omega z}\right)^{\nu-\lambda+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{\omega z})^{2k}}{\Gamma(k+\lambda+1)\,\Gamma(\mu k+\nu+1)},$$

which completes the proof of (21).

In particular, if $\omega = 1$ then $J_{\nu}^{\mu}(\omega z) = J_{\nu}^{\mu}(z)$ and this implies the validity of the equality (22). □

Corollary 3 *Let $\lambda, \nu, z \in \mathbb{C}$; $Re(\lambda) > 0$. Then the Riemann–Liouville fractional integral $R^{\lambda}[C_{\nu}(\omega z)]$ exists and the following relations hold true*

$$R^{\lambda}[C_{\nu}(\omega z)] = \frac{z^{(\lambda-\nu)/2}}{\omega^{(\lambda+\nu)/2}} J^{1}_{\nu-\lambda,\lambda}(2\sqrt{\omega z}) \quad (|\arg z| < \pi,\ \omega > 0), \tag{23}$$

and (*in particular*)

$$R^{\lambda}[C_{\nu}(z)] = z^{(\lambda-\nu)/2} J^{1}_{\nu-\lambda,\lambda}(2\sqrt{z}) \quad (|\arg z| < \pi). \tag{24}$$

Proof Analogically to the proofs of the previous corollaries the result immediately follows taking $\mu = 1$ in (21), namely

$$R^{\lambda}[C_{\nu}(\omega z)] = R^{\lambda}[J^{1}_{\nu}(\omega z)] = \frac{z^{(\lambda-\nu)/2}}{\omega^{(\lambda+\nu)/2}} J^{1}_{\nu-\lambda,\lambda}(2\sqrt{\omega z}) \quad (|\arg z| < \pi),$$

which is exactly the relation (23), the relation (24) follows taking additionally $\omega = 1$. □

Finally we can summarize that there are simple relations between the 3-index generalized Bessel–Maitland functions (4) and the corresponding fractional Riemann–Liouville integrals and derivatives of the 2-index Bessel–Maitland functions $J_{\nu}^{\mu}(z)$. More precisely these integrals and derivatives are expressed by the 3-index functions (4), up to matching power functions. Besides, the derivatives of integer order n of (3) can also be expressed by a function of the kind (3), multiplied by the constant $(-1)^n$.

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